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On morphisms of association schemes

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ABSTRACT

In this paper we present a necessary and sufficient condition under which a (combinatorial) morphism between association schemes induces an algebra homomorphism between their scheme rings. We will also discuss scheme ring homomorphisms of naturally valenced association schemes.

1. Introduction

Zieschang [Z, Chapter 5] introduced the notion of (combinatorial) morphisms for association schemes. Although morphisms of association schemes have very nice properties and play an important role in the study of algebraic structures of association schemes, they may not induce algebra homomorphisms between the scheme rings of association schemes. The main purpose of this article is to present a necessary and sufficient condition under which a morphism between association schemes induces an algebra homomorphism between their scheme rings. To do this, we will also need to study scheme ring homomorphisms of naturally valenced association schemes.

Let us first state some necessary definitions. Most of our notation and terminology stems from the book of Zieschang [Z]. Let X be a set, and S a partition of $X \times X$. Then S is called an *association scheme* on X if the following properties hold:

(i) $1_X \in S$, where $1_X := \{(x,x) \mid x \in X\}$. (Usually we simply denote 1_X by 1.)
(ii) For any $s \in S$, s^* is also in S, where $s^* := \{(y,z) \mid (z,y) \in s\}$.
(iii) For any $p,q,r \in S$, there exists a cardinal number a_{pqr} such that for any $(y,z) \in r$, $|\{x \in X \mid (y,x) \in p \text{ and } (x,z) \in q\}| = a_{pqr}$.

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Let (X, S) be an association scheme. For any $s \in S$, the *valency* of s is defined by $n_s := a_{ss^*1}$, and for any nonempty subset T of S, the *valency* of T is defined by $n_T := \sum_{s\in T} n_s$. A nonempty subset T of S is called *naturally valenced* ([Z], p. 63) if every element of T has finite valency.

Definition 1.1. (See [Z, p. 83].) Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be association schemes. A (*combinatorial*) *morphism* from (X, S) to $(\widetilde{X}, \widetilde{S})$ is a map $\phi : X \cup S \to \widetilde{X} \cup \widetilde{S}$ such that

(i) $\phi(X) \subseteq \widetilde{X}$ and $\phi(S) \subseteq \widetilde{S}$, and
(ii) for any $x, y \in X$ and $s \in S$ with $(x, y) \in s$, $(\phi(x), \phi(y)) \in \phi(s)$.

Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of association schemes. Then $\phi(1_X) = 1_{\widetilde{X}}$, and for any $s \in S$, $\phi(s^*) = \phi(s)^*$. The kernel of ϕ is defined by

$$\ker\phi := \{s \in S \mid \phi(s) = 1_{\widetilde{X}}\}.$$

Furthermore, if ϕ is bijective, then we say that ϕ is an *isomorphism*, (X, S) and $(\widetilde{X}, \widetilde{S})$ are *isomorphic*, and denote $(X, S) \cong (\widetilde{X}, \widetilde{S})$.

Let S be an association scheme on a set X. Then for any $p, q \in S$, let $pq := \{r \in S \mid a_{pqr} \neq 0\}$, and for any nonempty subsets P and Q of S, let $PQ := \{r \in S \mid \text{there exists } p \in P \text{ and } q \in Q \text{ such that } a_{pqr} \neq 0\}$. For any $s \in S$ and any nonempty subset P of S, we will write $\{s\}P$ as sP, and $P\{s\}$ as Ps. Let T be a nonempty subset of S. Then T is called a *closed subset* of S if $T^*T \subseteq T$, where $T^* = \{t^* \mid t \in T\}$. If X is a finite set, then T is a closed subset of S if and only if $TT \subseteq T$. If T is a closed subset of S, and for any $s \in S$, $sT = Ts$, then T is called a *normal closed subset* of S. For any morphism $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ of association schemes, $\ker\phi$ is a closed subset of S; but $\ker\phi$ may not be normal.

Let (X, S) be an association scheme, and T a closed subset of S. For any $x \in X$, let $xT := \{y \in X \mid (x, y) \in s \text{ for some } s \in T\}$. Define $t_{xT} := t \cap (xT \times xT)$, $\forall t \in T$, and $T_{xT} := \{t_{xT} \mid t \in T\}$. Then T_{xT} is an association scheme on xT (see [Z, Theorem 2.1.8]), called the *subscheme of S defined by xT* (see [Z, p. 22]). Furthermore, let $X/T := \{xT \mid x \in X\}$. For any $s \in S$, define $s^T := \{(yT, zT) \mid (y, z) \in s\}$, and define $S//T := \{s^T \mid s \in S\}$. If S is naturally valenced and T is a finite closed subset of S, then $S//T$ is an association scheme on X/T (see [Z, Theorem 4.1.3]), called the *quotient scheme of S over T* (see [Z, p. 65]).

For the rest of the paper we will always assume that the association schemes in the discussion are naturally valenced. Scheme rings are defined for association schemes of finite valencies in [Z, Section 9.1]. But this definition also works for naturally valenced association schemes. Let (X, S) be a naturally valenced association scheme. Let $\mathbb{C}$ be the field of complex numbers, and $\mathbb{C}X$ the $\mathbb{C}$-space with basis X. For any $s \in S$, similar to [Z, Section 9.1], define $\sigma_s \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}X)$ by

$$\sigma_s(x) := \sum_{(y,x)\in s} y, \quad \forall x \in X.$$

Note that since S is naturally valenced, σ_s is well defined. Furthermore, for any $p, q, r \in S$, a_{pqr} is finite, and similar to [Z, Lemma 9.1.1(i)], $\sigma_p\sigma_q = \sum_{r\in S} a_{pqr}\sigma_r = \sum_{r\in pq} a_{pqr}\sigma_r$. For any nonempty subset T of S, let $\sigma_T := \{\sigma_s \mid s \in T\}$, and $\mathbb{C}T$ the $\mathbb{C}$-space with basis σ_T. Then $\mathbb{C}S$ is a subring of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}X)$, called the *scheme ring* of the association scheme (X, S). Note that σ_1 is the identity element of $\mathbb{C}S$. For any closed subset T of S, $\mathbb{C}T$ is a subring of $\mathbb{C}S$, called a *scheme subring* of $\mathbb{C}S$.

Definition 1.2. Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a morphism. If there is an algebra homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ such that

$$\varphi(\sigma_s) = \frac{n_s}{n_{\phi(s)}}\sigma_{\phi(s)}, \quad \forall s \in S,$$

then we say that φ is *induced* by ϕ. We also say that ϕ induces an algebra homomorphism from scheme rings $\mathbb{C}S$ to $\mathbb{C}\widetilde{S}$.

Note that a morphism from naturally valenced association schemes (X,S) to $(\widetilde{X},\widetilde{S})$ may not induce an algebra homomorphism from scheme rings $\mathbb{C}S$ to $\mathbb{C}\widetilde{S}$. However, it is well known that an isomorphism between (naturally valenced) association schemes always induces an isomorphism between their scheme rings.

Now we introduce the concept of a scheme ring homomorphism.

Definition 1.3. Let (X,S) and $(\widetilde{X},\widetilde{S})$ be naturally valenced association schemes, and let T, $\widetilde{T}$ be closed subsets of S and $\widetilde{S}$, respectively. A map $\varphi:\mathbb{C}T\to\mathbb{C}\widetilde{T}$ is called a *scheme ring homomorphism* if the following properties hold:

(i) $\varphi:\mathbb{C}T\to\mathbb{C}\widetilde{T}$ is an algebra homomorphism.
(ii) $\varphi(\sigma_1)=\sigma_{\tilde{1}}$, the identity element of $\mathbb{C}\widetilde{S}$, and for any $s\in T$, $\varphi(\sigma_s)$ is a positive scalar multiple of some element in $\sigma_{\widetilde{T}}$.

Definition 1.3 is similar to the definition of a C-algebra homomorphism in [BI, p. 149]. In their book [BI], Bannai and Ito used C-algebras as a tool to study the algebraic properties of association schemes. For a commutative association scheme (X,S) of finite valency, the scheme ring $\mathbb{C}S$ together with the distinguished basis σ_S is a C-algebra. Another similar definition of a C-algebra homomorphism can be found in [B].

Let (X,S), $(\widetilde{X},\widetilde{S})$ be naturally valenced association schemes, and T, $\widetilde{T}$ closed subsets of S and $\widetilde{S}$, respectively. Let $\varphi:\mathbb{C}T\to\mathbb{C}\widetilde{T}$ be a scheme ring homomorphism. If φ is injective (surjective, bijective, resp.), then we say that φ is a *scheme ring monomorphism* (*epimorphism*, *isomorphism*, resp.). If φ is a scheme ring isomorphism, then we say that $\mathbb{C}T$ and $\mathbb{C}\widetilde{T}$ are *scheme ring isomorphic*, and denote $\mathbb{C}T\cong_s\mathbb{C}\widetilde{T}$. If φ is an isomorphism such that $\varphi(\sigma_T):=\{\varphi(\sigma_s)\mid s\in T\}=\sigma_{\widetilde{T}}$, then we say that φ is an *exact scheme ring isomorphism*, $\mathbb{C}T$ and $\mathbb{C}\widetilde{T}$ are *exactly isomorphic*, and denote $\mathbb{C}T\cong_x\mathbb{C}\widetilde{T}$. If both T and $\widetilde{T}$ are of finite valencies and $\mathbb{C}T\cong_s\mathbb{C}\widetilde{T}$, then $\mathbb{C}T\cong_x\mathbb{C}\widetilde{T}$ by Corollary 2.5 in Section 2.

One of our main results of the paper is the following

Theorem 1.4. *Let (X,S) and $(\widetilde{X},\widetilde{S})$ be naturally valenced association schemes, and $\phi:(X,S)\to(\widetilde{X},\widetilde{S})$ a morphism such that $\ker\phi$ is a finite subset of S. Then ϕ induces a scheme ring homomorphism $\varphi:\mathbb{C}S\to\mathbb{C}\widetilde{S}$ if and only if the following two conditions are satisfied.*

(i) *$\ker\phi$ is a normal closed subset of S.*
(ii) *For any $s\in S$, $n_{\phi(s)}=n_{s^T}$, where $T=\ker\phi$.*

We will prove Theorem 1.4 in Section 3. Theorem 1.4 has a few interesting corollaries. We will present these corollaries in Section 3. Let (X,S) and $(\widetilde{X},\widetilde{S})$ be naturally valenced association schemes, and $\phi:(X,S)\to(\widetilde{X},\widetilde{S})$ a morphism such that $\ker\phi$ is a finite subset of S. Then ϕ induces a morphism $\bar{\phi}:(X/\ker\phi, S//\ker\phi)\to(\widetilde{X},\widetilde{S})$ (see Section 3 below for the details). However, $\bar{\phi}$ is not necessarily injective. (See Example 3.11 in Section 3 below for such an example.) If for any $s\in S$, $n_{\phi(s)}=n_{s^T}$, where $T=\ker\phi$, then $\bar{\phi}$ is injective (see Lemma 3.1(iii) in Section 3). The next corollary can be regarded as a generalization of this result.

Corollary 1.5. *Let (X,S), $(\widetilde{X},\widetilde{S})$ be naturally valenced association schemes, and $\phi:(X,S)\to(\widetilde{X},\widetilde{S})$ a morphism such that $\ker\phi$ is a finite subset of S. Let $T=\ker\phi$. Then the following are equivalent.*

(i) *For any $s\in S$, $n_{\phi(s)}=n_{s^T}$.*
(ii) *$\phi(S)$ is a closed subset of $\widetilde{S}$, and for any $x\in X$, ϕ induces an isomorphism*

$$\bar{\phi}':(X/T,S//T)\cong\big(\phi(x)\phi(S),\phi(S)_{\phi(x)\phi(S)}\big)$$

such that $\bar{\phi}'(yT) = \phi(y)$ *for any* $yT \in X/T$*, and* $\bar{\phi}'(s^T) = \phi(s)_{\phi(x)\phi(S)}$ *for any* $s^T \in S//T$.

Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes such that $\ker\phi$ is a finite subset of S and $\phi(S)$ is a closed subset of $\widetilde{S}$. Example 3.10 in Section 3 indicates that it is not necessarily true that $(X/\ker\phi, S//\ker\phi) \cong (\phi(x)\phi(S), \phi(S)_{\phi(x)\phi(S)})$ for some $x \in X$.

In Section 2 we study scheme ring homomorphisms. Using the properties obtained in Section 2, we will prove Theorem 1.4 and its corollaries in Section 3.

2. Scheme ring homomorphisms

In this section we study the basic properties of scheme ring homomorphisms. Although these properties hold for the common algebraic structures, they are not trivial for the scheme rings of naturally valenced association schemes.

Let (X, S) be a naturally valenced association scheme. If S is of finite valency, then the scheme ring $\mathbb{C}S$ and the Bose–Mesner algebra of (X, S) are identical. As generalizations of scheme rings and Bose–Mesner algebras of association schemes of finite valencies, table algebras have interesting applications to association schemes. For the definition of a table algebra, the reader is referred to [BZ, Definition 1.1]. For any table algebra $(A, \mathbf{B})$, there is a unique algebra homomorphism $|| : A \to \mathbb{C}$ such that $|b| > 0$ for all $b \in \mathbf{B}$ (see Proposition 3.12 and Theorem 3.14 of [AFM]). This algebra homomorphism $|| : A \to \mathbb{C}$ is called the *degree map* of $(A, \mathbf{B})$. Note that for any $b \in \mathbf{B}$, $|b| = |b^*|$ (see Theorem 3.14 of [AFM]). A table algebra $(A, \mathbf{B})$ is called *standard* if for all $b \in \mathbf{B}$, $|b| = \lambda_{bb^*1}$ (see Definition 1.2 of [BZ]). Let (X, S) be a naturally valenced association scheme, and T a finite closed subset of S. Then the scheme subring $\mathbb{C}T$ together with its distinguished basis σ_T is a standard table algebra, and its degree map is defined by $\sigma_s \mapsto n_s$, $\forall s \in T$. In particular, if (X, S) is an association scheme of finite valency, then the scheme ring $\mathbb{C}S$ together with the distinguished basis σ_S is a standard table algebra.

The next lemma is an immediate consequence of [AFM, Theorem 3.14].

Lemma 2.1. *Let* (X, S) *be a naturally valenced association scheme, and* T *a finite closed subset of* S*. Then for any* $s \in T$*,* $n_s = n_{s^*}$.

The next easy lemma is very useful.

Lemma 2.2. *Let* (X, S) *and* $(\widetilde{X}, \widetilde{S})$ *be naturally valenced association schemes,* T *and* $\widetilde{T}$ *closed subsets of* S *and* $\widetilde{S}$*, respectively, and* $\varphi : \mathbb{C}T \to \mathbb{C}\widetilde{T}$ *a scheme ring homomorphism. If* T *is a finite closed subset of* S*, then for any* $s \in T$*, there is* $\tilde{s} \in \widetilde{T}$ *such that*

$$\varphi(\sigma_s) = \frac{n_s}{n_{\tilde{s}}}\sigma_{\tilde{s}}.$$

Proof. By [Z, Lemma 1.1.3(iv)], the map $\tilde{\nu} : \mathbb{C}\widetilde{T} \to \mathbb{C}$ defined by $\tilde{\nu}(\sigma_{\tilde{s}}) = n_{\tilde{s}}$, for any $\tilde{s} \in \widetilde{T}$, is an algebra homomorphism. Assume that T is a finite closed subset of S. Then $(\mathbb{C}T, \sigma_T)$ is a table algebra, and $\tilde{\nu}\varphi : \mathbb{C}T \to \mathbb{C}$ is a degree map. Note that $\nu : \mathbb{C}T \to \mathbb{C}$ defined by $\nu(\sigma_s) = n_s$, for any $s \in T$, is also a degree map. But $(\mathbb{C}T, \sigma_T)$ has a unique degree map by [AFM, Theorem 3.14]. So $\nu = \tilde{\nu}\varphi$. Hence the lemma holds. □

Let (X, S), $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ a scheme ring homomorphism. For any nonempty subset T of S, define

$$\varphi(T) := \{\tilde{s} \in \widetilde{S} \mid \sigma_{\tilde{s}} \text{ is a positive scalar multiple of } \varphi(\sigma_s) \text{ for some } s \in T\},$$

and for any nonempty subset $\widetilde{T}$ of $\widetilde{S}$, define

$$\varphi^{-1}(\widetilde{T}) := \{s \in S \mid \varphi(\sigma_s) \text{ is a positive scalar multiple of } \sigma_{\tilde{s}} \text{ for some } \tilde{s} \in \widetilde{T}\}.$$

Lemma 2.3. *Let (X,S), $(\widetilde{X},\widetilde{S})$ be naturally valenced association schemes, and $\varphi:\mathbb{C}S\to\mathbb{C}\widetilde{S}$ a scheme ring homomorphism. Then the following hold.*

(i) *For any $s\in S$, if $\varphi(\sigma_s)$ is a positive scalar multiple of $\sigma_{\tilde{s}}$ for some $\tilde{s}\in\widetilde{S}$, then $\varphi(\sigma_{s^*})$ is a positive scalar multiple of $\sigma_{\tilde{s}^*}$.*
(ii) *If T is a closed subset of S, then $\varphi(T)$ is a closed subset of $\widetilde{S}$.*
(iii) *If $\widetilde{T}$ is a closed subset of $\widetilde{S}$, then $\varphi^{-1}(\widetilde{T})$ is a closed subset of S.*

Proof. (i) Assume that $\varphi(\sigma_s)=\lambda_s\sigma_{\tilde{s}}$ and $\varphi(\sigma_{s^*})=\lambda_{s^*}\sigma_{\tilde{t}}$, where λ_s,λ_{s^*} are positive real numbers, and $\tilde{s},\tilde{t}\in\widetilde{S}$. Then $\varphi(\sigma_1)=\sigma_{\tilde{1}}$ implies that

$$\lambda_s\lambda_{s^*}\sigma_{\tilde{s}}\sigma_{\tilde{t}}=\varphi(\sigma_s\sigma_{s^*})=n_s\sigma_{\tilde{1}}+\sum_{r\in S\setminus\{1\}}a_{ss^*r}\varphi(\sigma_r).$$

Thus, $a_{\tilde{s}\tilde{t}\tilde{1}}\neq 0$. So $\tilde{t}=\tilde{s}^*$, and (i) holds.

(ii) Let $\tilde{s},\tilde{t}\in\varphi(T)$. Then there are $s,t\in T$ such that $\varphi(\sigma_s)=\lambda_s\sigma_{\tilde{s}}$ and $\varphi(\sigma_t)=\lambda_t\sigma_{\tilde{t}}$, where λ_s,λ_t are positive real numbers. By (i) we see that $\varphi(\sigma_{s^*})=\lambda_{s^*}\sigma_{\tilde{s}^*}$ for some positive real number λ_{s^*}. But T is closed. So $s^*t\subseteq T$. Hence

$$\lambda_{s^*}\lambda_t\sum_{\tilde{r}\in\widetilde{S}}a_{\tilde{s}^*\tilde{t}\tilde{r}}\sigma_{\tilde{r}}=\lambda_{s^*}\lambda_t\sigma_{\tilde{s}^*}\sigma_{\tilde{t}}=\varphi(\sigma_{s^*}\sigma_t)=\sum_{r\in T}a_{s^*tr}\varphi(\sigma_r).$$

Thus, $\tilde{s}^*\tilde{t}\subseteq\varphi(T)$. Therefore, $\varphi(T)$ is a closed subset of $\widetilde{S}$.

The proof of (iii) is similar to the proof of (ii). □

The next lemma gives a sufficient condition under which a scheme ring isomorphism is an exact isomorphism. This lemma will be needed in the next section.

Lemma 2.4. *Let (X,S), $(\widetilde{X},\widetilde{S})$ be naturally valenced association schemes, T and $\widetilde{T}$ closed subsets of S and $\widetilde{S}$, respectively, and $\varphi:\mathbb{C}T\to\mathbb{C}\widetilde{T}$ a scheme ring monomorphism. If for any $s\in T$, there is $\tilde{s}\in\widetilde{T}$ such that $\varphi(\sigma_s)=(n_s/n_{\tilde{s}})\sigma_{\tilde{s}}$, then $\mathbb{C}T\cong_x\mathbb{C}(\varphi(T))$.*

Proof. Let $s\in T$ such that $\varphi(\sigma_s)=(n_s/n_{\tilde{s}})\sigma_{\tilde{s}}$ for some $\tilde{s}\in\widetilde{S}$. Then by Lemma 2.3(i), $\varphi(\sigma_{s^*})$ is a scalar multiple of $\sigma_{\tilde{s}^*}$. So $\varphi(\sigma_{s^*})=(n_{s^*}/n_{\tilde{s}^*})\sigma_{\tilde{s}^*}$ by the assumption of the lemma. Thus, $\varphi(\sigma_{s^*}\sigma_s)=\varphi(\sigma_{s^*})\varphi(\sigma_s)$ yields that

$$n_{s^*}\sigma_{\tilde{1}}+\sum_{t\in T\setminus\{1\}}a_{s^*st}\varphi(\sigma_t)=\frac{n_{s^*}n_s}{n_{\tilde{s}^*}n_{\tilde{s}}}\Bigg(n_{\tilde{s}^*}\sigma_{\tilde{1}}+\sum_{\tilde{t}\in\widetilde{S}\setminus\{\tilde{1}\}}a_{\tilde{s}^*\tilde{s}\tilde{t}}\,\sigma_{\tilde{t}}\Bigg).\tag{1}$$

But φ is injective. So for any $t\in T\setminus\{1\}$, $\varphi(\sigma_t)$ is not a positive scalar multiple of $\sigma_{\tilde{1}}$. Thus, from (1) we get that $n_s=n_{\tilde{s}}$. Hence, for any $s\in S$, we have proved that $\varphi(\sigma_s)=\sigma_{\tilde{s}}$ for some $\tilde{s}\in\widetilde{S}$. So

$$\mathbb{C}T\longrightarrow\mathbb{C}\big(\varphi(T)\big),\qquad\sigma_s\longmapsto\varphi(\sigma_s)$$

is an exact scheme ring isomorphism. □

The next corollary is a direct consequence of Lemmas 2.2 and 2.4.

Corollary 2.5. *Let (X,S) and $(\widetilde{X},\widetilde{S})$ be naturally valenced association schemes, and $\varphi:\mathbb{C}S\to\mathbb{C}\widetilde{S}$ a scheme ring homomorphism. Let T be a finite closed subset of S such that the restriction of φ to $\mathbb{C}T$ is injective. Then $\mathbb{C}T\cong_x\mathbb{C}(\varphi(T))$, and $n_T=n_{\varphi(T)}$.*

Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be association schemes of finite valencies. If $\mathbb{C}S \cong_s \mathbb{C}\widetilde{S}$, then by Corollary 2.5, $\mathbb{C}S \cong_x \mathbb{C}\widetilde{S}$, and $n_S = n_{\widetilde{S}}$.

Let (X, S), $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ a scheme ring homomorphism. Then define

$$\ker_S(\varphi) := \{s \in S \mid \varphi(\sigma_s) \text{ is a positive scalar multiple of } \sigma_{\tilde{1}}\}.$$

That is, $\ker_S(\varphi) = \varphi^{-1}(\{\tilde{1}\})$. So $\ker_S(\varphi)$ is a closed subset of S by Lemma 2.3(iii).

The next theorem is one of the main results in this section. A result similar to Theorem 2.6(i) below was proved for table algebras in [X2, Theorem 3.1(i)]. (Also see [X3, Proposition 3.5 and Theorem 4.1].) The author is indebted to H. Blau for pointing out a better proof of [X2, Theorem 3.1(i)], which leads to the proof of Theorem 2.6(i) below.

Theorem 2.6. *Let (X, S), $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ a scheme ring homomorphism. Then the following hold.*

(i) $\ker_S(\varphi)$ *is a normal closed subset of S.*
(ii) *Let $r, s \in S$. Then $r\ker_S(\varphi) = s\ker_S(\varphi)$ if and only if $\varphi(\sigma_r)$ and $\varphi(\sigma_s)$ are scalar multiples of the same element in $\sigma_{\widetilde{S}}$.*
(iii) *φ is injective if and only if* $\ker_S(\varphi) = \{1\}$.

Proof. (i) Since $\ker_S(\varphi)$ is a closed subset of S by Lemma 2.3(iii), we only need to show that for any $s \in S$, $s\ker_S(\varphi) = \ker_S(\varphi)s$. Assume that $\varphi(\sigma_s) = \lambda_s\sigma_{\tilde{s}}$ for some $\tilde{s} \in \widetilde{S}$, where λ_s is a positive real number. Then we prove that $\varphi^{-1}(\{\tilde{s}\}) = s\ker_S(\varphi)$. Let $t \in \ker_S(\varphi)$. Then $\varphi(\sigma_t) = \lambda_t\sigma_{\tilde{1}}$ for some positive real number λ_t. Hence,

$$\lambda_s\lambda_t\sigma_{\tilde{s}} = \lambda_t\varphi(\sigma_s) = \varphi(\sigma_s\sigma_t) = \sum_{r\in st} a_{str}\varphi(\sigma_r).$$

So $st \subseteq \varphi^{-1}(\{\tilde{s}\})$. Hence $s\ker_S(\varphi) \subseteq \varphi^{-1}(\{\tilde{s}\})$. On the other hand, let $r \in \varphi^{-1}(\{\tilde{s}\})$. Then $\varphi(\sigma_r) = \lambda_r\sigma_{\tilde{s}}$ for some positive real number λ_r. Since $\varphi(\sigma_{s^*}) = \lambda_{s^*}\sigma_{\tilde{s}^*}$ for some positive real number λ_{s^*} by Lemma 2.3(i), we see that

$$\varphi(\sigma_{s^*}\sigma_r) = \lambda_{s^*}\lambda_r\sigma_{\tilde{s}^*}\sigma_{\tilde{s}} = \lambda_{s^*}\lambda_r\left(n_{\tilde{s}^*}\sigma_{\tilde{1}} + \sum_{\tilde{t}\in\widetilde{S}\setminus\{\tilde{1}\}} a_{\tilde{s}^*\tilde{s}\tilde{t}}\sigma_{\tilde{t}}\right).$$

So there exists $t \in \ker_S(\varphi)$ such that $t \in s^*r$. Thus, $a_{s^*rt} \neq 0$. So $a_{r^*st^*} \neq 0$ by [Z, Lemma 1.1.1(ii)]. But $a_{r^*st^*}n_{t^*} = a_{t^*s^*r^*}n_{r^*}$ by [Z, Lemma 1.1.3(ii)]. So $a_{t^*s^*r^*} \neq 0$. Hence $a_{str} \neq 0$ by [Z, Lemma 1.1.1(ii)]. Thus, $r \in st \subseteq s\ker_S(\varphi)$. Hence, $\varphi^{-1}(\{\tilde{s}\}) \subseteq s\ker_S(\varphi)$. Therefore, we have proved that $\varphi^{-1}(\{\tilde{s}\}) = s\ker_S(\varphi)$.

Similarly we can prove that $\varphi^{-1}(\{\tilde{s}\}) = \ker_S(\varphi)s$. Hence, for any $s \in S$, $s\ker_S(\varphi) = \ker_S(\varphi)s$. Thus $\ker_S(\varphi)$ is a normal closed subset of S, and (i) holds.

(ii) Assume that $\varphi(\sigma_s) = \lambda_s\sigma_{\tilde{s}}$ for some $\tilde{s} \in \widetilde{S}$ and $\varphi(\sigma_r) = \lambda_r\sigma_{\tilde{r}}$ for some $\tilde{r} \in \widetilde{S}$, where λ_s and λ_r are positive real numbers. Then $s\ker_S(\varphi) = \varphi^{-1}(\{\tilde{s}\})$ and $r\ker_S(\varphi) = \varphi^{-1}(\{\tilde{r}\})$ by the proof of (i). Hence, $r\ker_S(\varphi) = s\ker_S(\varphi)$ if and only if $\varphi^{-1}(\{\tilde{r}\}) = \varphi^{-1}(\{\tilde{s}\})$ if and only if $\tilde{r} = \tilde{s}$. So (ii) holds.

(iii) If φ is injective, then clearly $\ker_S(\varphi) = \{1\}$. Now assume that $\ker_S(\varphi) = \{1\}$. Let $r, s \in S$ such that $r \neq s$. Then $r\ker_S(\varphi) \neq s\ker_S(\varphi)$. Hence, $\varphi(\sigma_r)$ and $\varphi(\sigma_s)$ are not scalar multiples of the same element in $\sigma_{\widetilde{S}}$ by (ii). Thus, $\{\varphi(\sigma_s) \mid s \in S\}$ is a linearly independent subset of $\mathbb{C}\widetilde{S}$. So φ is injective, and (iii) holds. $\square$

Let (X, S) be a naturally valenced association scheme. For any finite closed subset T of S, although we have the *canonical morphism* (or *natural homomorphism*, as called in [Z])

$$\phi : (X,S) \longrightarrow (X/T, S//T), \qquad \phi(x) = xT, \quad \forall x \in X, \qquad \phi(s) = s^T, \quad \forall s \in S$$

by [Z, Theorem 5.3.1], we may not have a scheme ring homomorphism from $\mathbb{C}S$ to $\mathbb{C}(S//T)$. Hanaki proved that if S is of finite valency and T is a normal closed subset of S, then the canonical morphism $\phi : (X,S) \to (X/T, S//T)$ induces a scheme ring epimorphism $\pi : \mathbb{C}S \to \mathbb{C}(S//T)$ (see [H, Theorem 3.4] and [Z, Lemma 9.3.3]). Xu [X1, Corollary 2.2] proved that if S is of finite valency and T is a closed subset of S such that the canonical morphism $\phi : (X,S) \to (X/T, S//T)$ induces a scheme ring epimorphism $\pi : \mathbb{C}S \to \mathbb{C}(S//T)$, then T is a normal closed subset of S. More generally, we have the following

Corollary 2.7. *Let (X,S) be a naturally valenced association scheme, and T a finite closed subset of S. Then the canonical morphism $\phi : (X,S) \to (X/T, S//T)$ induces a scheme ring epimorphism*

$$\pi : \mathbb{C}S \longrightarrow \mathbb{C}(S//T), \qquad \sigma_s \longmapsto \frac{n_s}{n_{s^T}} \sigma_{s^T}$$

if and only if T is a normal closed subset.

Proof. If T is a normal closed subset, then Zieschang's proof of [Z, Lemma 9.3.3] also shows that π is a scheme ring epimorphism. On the other hand, if π is a scheme ring epimorphism, then $T = \ker_S(\pi)$ is a normal closed subset by Theorem 2.6(i). □

Let (X,S) be a naturally valenced association scheme, and T a finite normal closed subset of S. Then the scheme ring epimorphism $\pi : \mathbb{C}S \to \mathbb{C}(S//T)$ in Corollary 2.7 is called the *canonical epimorphism*.

Let (X,S) be a naturally valenced association scheme, and P a nonempty finite subset of S. Then define $(\sigma_P)^+ := \sum_{s \in P} \sigma_s$. Furthermore, for any $p, q \in S$, define $a_{pTq} := \sum_{s \in T} a_{psq}$ (see [Z, p. 24]). The next result generalizes Lemmas 3.1 and 3.2 of [H].

Lemma 2.8. *Let (X,S) be a naturally valenced association scheme, and T a finite closed subset of S. Then for any $s \in S$,*

$$\sigma_s (\sigma_T)^+ = \frac{n_s n_T}{n_{sT}} (\sigma_{sT})^+.$$

Furthermore, $a_{sTs} = n_s n_T / n_{sT}$.

Proof. For any $p \in sT$, $a_{sTs} = a_{sTp}$ by [Z, Lemma 2.3.1(i)]. So

$$\sigma_s (\sigma_T)^+ = \sum_{p \in sT} a_{sTp} \sigma_p = a_{sTs} \sum_{p \in sT} \sigma_p = a_{sTs} (\sigma_{sT})^+.$$

By [Z, Lemma 1.1.3(iv)], the map $\nu : \mathbb{C}S \to \mathbb{C}$ defined by $\nu(\sigma_s) = n_s$, for any $s \in S$, is an algebra homomorphism. Applying ν to the equation $\sigma_s (\sigma_T)^+ = a_{sTs} (\sigma_{sT})^+$, we get that $a_{sTs} = n_s n_T / n_{sT}$. So the lemma holds. □

The next theorem is our last main result of this section.

Theorem 2.9. *Let (X,S) and $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ a scheme ring homomorphism. If $\ker_S(\varphi)$ is a finite subset of S, then φ induces a scheme ring monomorphism*

$$\tilde{\varphi} : \mathbb{C}\big(S// \ker_S(\varphi)\big) \longrightarrow \mathbb{C}\widetilde{S}$$

such that $\varphi = \tilde{\varphi}\pi$, where $\pi : \mathbb{C}S \to \mathbb{C}(S//\ker_S(\varphi))$ is the canonical epimorphism. In particular, $\mathbb{C}(S//\ker_S(\varphi)) \cong_s \mathbb{C}(\varphi(S))$.

Proof. Let $T := \ker_S(\varphi)$. Then T is a normal closed subset of S by Theorem 2.6(i). In the following we first prove that the map

$$\tilde{\varphi} : \mathbb{C}\big(S//\ker_S(\varphi)\big) \longrightarrow \mathbb{C}\widetilde{S}$$

defined by

$$\tilde{\varphi}(\sigma_{s^T}) = \frac{n_{s^T}}{n_s}\varphi(\sigma_s), \quad \forall s \in S \tag{2}$$

is well defined. Since T is a finite closed subset of S, for any $t \in T$, $\varphi(\sigma_t) = n_t\sigma_{\tilde{1}}$ by Lemma 2.2. But T is normal. So for any $s \in S$, $sT = Ts$ and [Z, Lemma 4.1.3(iii)] yield that $n_{s^T} = n_T^{-1}n_{TsT} = n_T^{-1}n_{sT}$. Hence from Lemma 2.8, we see that $\sigma_s(\sigma_T)^+ = (n_s/n_{s^T})(\sigma_{sT})^+$, for any $s \in S$. Thus, applying φ to the equation $\sigma_s(\sigma_T)^+ = (n_s/n_{s^T})(\sigma_{sT})^+$, we get that

$$n_T\varphi(\sigma_s) = \frac{n_s}{n_{s^T}}\varphi\big((\sigma_{sT})^+\big). \tag{3}$$

Let $p, q \in S$. Since T is normal, $p^T = q^T$ if and only if $pT = qT$ by [Z, Lemma 4.1.1]. Hence, (3) yields that

$$\frac{n_{p^T}}{n_p}\varphi(\sigma_p) = \frac{n_{q^T}}{n_q}\varphi(\sigma_q) \quad \text{if } p^T = q^T. \tag{4}$$

Thus, $\tilde{\varphi}$ is well defined.

Now we show that $\tilde{\varphi}$ is an algebra homomorphism. For any $s \in S$ and any $w \in sT$, (4) yields that

$$\varphi(\sigma_w) = \frac{n_w}{n_{w^T}}\frac{n_{s^T}}{n_s}\varphi(\sigma_s).$$

So for any $p^T, q^T \in S//T$, from (2) we get that

$$\begin{aligned}
\tilde{\varphi}(\sigma_{p^T})\tilde{\varphi}(\sigma_{q^T}) &= \frac{n_{p^T}}{n_p}\frac{n_{q^T}}{n_q}\sum_{w\in S} a_{pqw}\varphi(\sigma_w) \\
&= \sum_{s^T\in S//T}\left(\frac{n_{p^T}}{n_p}\frac{n_{q^T}}{n_q}\sum_{w\in sT} a_{pqw}\varphi(\sigma_w)\right) \\
&= \sum_{s^T\in S//T}\left[\frac{n_{p^T}}{n_p}\frac{n_{q^T}}{n_q}\sum_{w\in sT}\left(a_{pqw}\frac{n_w}{n_{w^T}}\right)\right]\frac{n_{s^T}}{n_s}\varphi(\sigma_s).
\end{aligned} \tag{5}$$

Note that for any $s \in S$, $a_{sTs} = n_s/n_{s^T}$ by Lemma 2.8. So for any $p^T, q^T, s^T \in S//T$, Theorem 4.1.3(ii) and Lemma 2.5.4(iii) of [Z], and Lemma 2.8 yield that

$$a_{p^Tq^Ts^T} = n_T^{-1}\sum_{u\in pT}\sum_{v\in qT} a_{uvs} = \frac{n_{p^T}}{n_p}\frac{n_{q^T}}{n_q}\sum_{w\in sT}\left(a_{pqw}\frac{n_w}{n_{w^T}}\right).$$

So from (5) we see that

$$\tilde{\varphi}(\sigma_{p^T})\tilde{\varphi}(\sigma_{q^T}) = \sum_{s^T \in S//T} a_{p^T q^T s^T} \tilde{\varphi}(\sigma_{s^T}) = \tilde{\varphi}(\sigma_{p^T}\sigma_{q^T}).$$

Thus, $\tilde{\varphi}$ is an algebra homomorphism.

Clearly $\tilde{\varphi}$ is a scheme ring homomorphism. Furthermore, for any $s \in S$, $s^T \in \ker_{S//T}(\tilde{\varphi})$ if and only if $s \in \ker_S(\varphi)$ by (2). So $\ker_{S//T}(\tilde{\varphi}) = \{1^T\}$. Hence $\tilde{\varphi}$ is a scheme ring monomorphism. Clearly $\varphi = \tilde{\varphi}\pi$. $\square$

Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ a scheme ring homomorphism such that $\ker_S(\varphi)$ is a finite subset of S. Then we say that the scheme ring monomorphism $\tilde{\varphi} : \mathbb{C}(S// \ker_S(\varphi)) \to \mathbb{C}\widetilde{S}$ in Theorem 2.9 *is induced by* φ.

3. Morphisms that induce scheme ring homomorphisms

In this section we will prove Theorem 1.4 and Corollary 1.5. We will also present a few other corollaries of Theorem 1.4. Let us prove a few lemmas first.

Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes. Then the restriction of ϕ to X is denoted by ϕ_X, and the restriction of ϕ to S is denoted by ϕ_S. Let $T = \ker\phi$. If T is a finite subset of S, then from [Z, Lemmas 5.1.5 and 5.1.4(i)],

$$\bar{\phi} : (X/T, S//T) \longrightarrow (\widetilde{X}, \widetilde{S}), \qquad xT \longmapsto \phi(x), \qquad s^T \longmapsto \phi(s)$$

is a morphism, called the *morphism induced by* ϕ. Note that $\bar{\phi}_{X/T}$ is injective by [Z, Lemma 5.1.4(i)]. However, $\bar{\phi}_{S//T}$ may not be injective. (See Example 3.11 below for such an example.) A sufficient condition under which $\bar{\phi}_{S//T}$ is injective is given in the next lemma. If ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$, then we will prove that $\bar{\phi}_{S//T}$ is injective (see Corollary 3.4(i) below).

Lemma 3.1. *Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes such that* $\ker\phi$ *is a finite subset of S. Let $T = \ker\phi$. Then the following hold.*

(i) *For any $p, q, r \in S$, $a_{\phi(p)\phi(q)\phi(r)} \geqslant a_{p^T q^T r^T}$.*
(ii) *For any $s \in S$, $n_{\phi(s)} \geqslant n_{s^T}$.*
(iii) *If $n_{s^T} = n_{\phi(s)}$ for all $s \in S$, then $\bar{\phi}_{S//T}$ is injective.*

Proof. Let us first assume that ϕ_X is injective. Let $p, q, r \in S$. Let $y, z \in X$ with $(y, z) \in r$. Then $(\phi(y), \phi(z)) \in \phi(r)$. Furthermore, for any $x \in X$ such that $(y, x) \in p$ and $(x, z) \in q$, we see that $(\phi(y), \phi(x)) \in \phi(p)$, and $(\phi(x), \phi(z)) \in \phi(q)$. But ϕ_X is injective. So ϕ induces an injective map

$$\bigl\{x \in X \bigm| (y, x) \in p \text{ and } (x, z) \in q\bigr\} \longrightarrow \bigl\{\tilde{x} \in \widetilde{X} \bigm| \bigl(\phi(y), \tilde{x}\bigr) \in \phi(p) \text{ and } \bigl(\tilde{x}, \phi(z)\bigr) \in \phi(q)\bigr\},$$
$$x \longmapsto \phi(x).$$

Thus, we have that

$$a_{pqr} \leqslant a_{\phi(p)\phi(q)\phi(r)}.$$

So (i) holds. In particular, for any $p \in S$, since $\phi(p^*) = \phi(p)^*$, (i) yields that $a_{pp^*1} \leqslant a_{\phi(p)\phi(p)^*1}$. That is,

$$n_p \leqslant n_{\phi(p)}, \qquad \text{for any } p \in S.$$

So (ii) holds. For any $\tilde{s} \in \widetilde{S}$, let $\phi^{-1}(\tilde{s}) = \{s \in S \mid \phi(s) = \tilde{s}\}$. Fix $x \in X$. For any $s \in S$, let

$$V_s = \{\phi(y) \in \widetilde{X} \mid y \in X \text{ and } (\phi(x), \phi(y)) \in \phi(s)\},$$

and

$$U_s = \{y \in X \mid (x, y) \in u \text{ for some } u \in \phi^{-1}(\phi(s))\}.$$

Then $\phi(U_s) = V_s$. Since ϕ_X is injective, $|U_s| = |V_s|$. So

$$n_{\phi(s)} \geqslant |V_s| = |U_s| = \sum_{u \in \phi^{-1}(\phi(s))} n_u. \tag{6}$$

Since ϕ_X injective implies that $T = \{1\}$, the hypothesis of (iii) yields that $n_s = n_{\phi(s)}$, for all $s \in S$. Since $s \in \phi^{-1}(\phi(s))$, (6) then forces that $\phi^{-1}(\phi(s)) = \{s\}$, for all $s \in S$. Hence ϕ_S is injective, and (iii) holds.

Now assume that ϕ is an arbitrary morphism. Since T is a finite closed subset of S, ϕ induces a morphism $\bar{\phi} : (X/T, S//T) \to (\widetilde{X}, \widetilde{S})$ such that $\bar{\phi}_{X/T}$ is injective. Note that for any $s \in S$, $\phi(s) = \bar{\phi}(s^T)$. Applying what we have just proved to $\bar{\phi}$, we see that (i), (ii), and (iii) hold. □

Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes such that $\ker\phi = \{1\}$. Example 3.11 below says that ϕ may not be a monomorphism. But if for all $s \in S$, $n_s = n_{\phi(s)}$, then ϕ is a monomorphism by Lemma 3.1(iii).

The next corollary is an immediate consequence of Lemma 3.1(ii), (iii).

Corollary 3.2. *Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes, where S is assumed to have finite valency. Let $T = \ker\phi$. Then the following hold.*

(i) *If $\bar{\phi}_{S//T}$ is injective, then $n_S/n_T \leqslant n_{\phi(S)}$.*
(ii) *The conditions that $\bar{\phi}_{S//T}$ is injective and $n_S/n_T = n_{\phi(S)}$ both hold if and only if $n_{\phi(s)} = n_{s^T}$ for any $s \in S$.*

Proof. For any $p, q \in S$, if $p^T = q^T$, then $\phi(p) = \phi(q)$ by [Z, Lemma 5.1.4(ii)]. Hence by Lemma 3.1(ii) and [Z, Theorem 4.1.3(iii)],

$$\sum_{s^T \in S//T} n_{\phi(s)} \geqslant \sum_{s^T \in S//T} n_{s^T} = n_{S//T} = n_S/n_T. \tag{7}$$

If $\bar{\phi}_{S//T}$ is injective, then for any $p, q \in S$ such that $\phi(p) = \phi(q)$, we have $p^T = q^T$. Thus,

$$\sum_{s^T \in S//T} n_{\phi(s)} = n_{\phi(S)} \quad \Leftrightarrow \quad \bar{\phi}_{S//T} \text{ is injective.} \tag{8}$$

So (i) follows from (7) and (8). If $\bar{\phi}_{S//T}$ is injective and $n_{\phi(S)} = n_S/n_T$, then (7), (8), and Lemma 3.1(ii) imply that $n_{\phi(s)} = n_{s^T}$ for all $s \in S$. This proves one direction of (ii). Conversely, if $n_{s^T} = n_{\phi(s)}$ for all $s \in S$, then $\bar{\phi}_{S//T}$ is injective by Lemma 3.1(iii). Thus, (7) and (8) imply that $n_{\phi(S)} = n_S/n_T$. So (ii) holds. □

Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes such that S has finite valency and $n_{\phi(S)} = n_S/n_T$, where $T = \ker\phi$. Example 3.11 below indicates that $\bar{\phi}_{S//T}$ may not be injective, and n_{s^T} may not equal $n_{\phi(s)}$ for some $s \in S$.

Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes. If ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$, then

$$\ker_S(\varphi) = \ker\phi \quad \text{and} \quad \varphi(S) = \phi(S). \tag{9}$$

Lemma 3.3. *Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes such that $\ker\phi$ is a finite subset of S. Assume that ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$. Let $T = \ker\phi$. Then the following hold.*

(i) *$T = \ker_S(\varphi)$ is a finite normal closed subset of S.*
(ii) *Let $\tilde{\varphi} : \mathbb{C}(S//\ker_S(\varphi)) \to \mathbb{C}\widetilde{S}$ be the scheme ring monomorphism induced by φ (see Theorem 2.9), and let $\bar{\phi} : (X/T, S//T) \to (\widetilde{X}, \widetilde{S})$ be the morphism induced by ϕ. Then $\bar{\phi}$ induces $\tilde{\varphi}$.*
(iii) *For any $s \in S$, $n_{\phi(s)} = n_{s^T}$.*

Proof. (i) $T = \ker_S(\varphi)$ by (9), and T is a normal closed subset of S by Theorem 2.6(i).

(ii) Since φ is induced by ϕ, for any $s \in S$, $\varphi(\sigma_s) = (n_s/n_{\phi(s)})\sigma_{\phi(s)}$ by Definition 1.2. Hence, for any $s^T \in S//T$, since $\tilde{\varphi}$ is induced by φ, from the proof of Theorem 2.9, we have that

$$\tilde{\varphi}(\sigma_{s^T}) = \frac{n_{s^T}}{n_s}\varphi(\sigma_s) = \frac{n_{s^T}}{n_{\phi(s)}}\sigma_{\phi(s)}.$$

Note that for any $s^T \in S//T$, $\bar{\phi}(s^T) = \phi(s)$. So $\tilde{\varphi}$ is also induced by $\bar{\phi}$, and (ii) holds.

(iii) Since $\tilde{\varphi}$ is injective, and for any $s^T \in S//T$, $\tilde{\varphi}(\sigma_{s^T}) = (n_{s^T}/n_{\phi(s)})\sigma_{\phi(s)}$ by (ii), from the proof of Lemma 2.4, we must have that $n_{s^T} = n_{\phi(s)}$, for any $s \in S$. So (iii) holds. □

The next corollary is a direct consequence of Lemmas 3.1(iii) and 3.3(iii).

Corollary 3.4. *Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes such that $\ker\phi$ is a finite subset of S. Assume that ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$. Let $\bar{\phi} : (X/\ker\phi, S//\ker\phi) \to (\widetilde{X}, \widetilde{S})$ be the morphism induced by ϕ. Then the following hold.*

(i) *$\bar{\phi}$ is a monomorphism.*
(ii) *ϕ is a monomorphism if and only if $\ker\phi = \{1\}$.*

Proof. (i) Let $T = \ker\phi$. By [Z, Lemma 5.1.4(i)], $\bar{\phi}_{X/T}$ is injective. But for any $s \in S$, $n_{s^T} = n_{\phi(s)}$ by Lemma 3.3(iii). So $\bar{\phi}_{S//T}$ is also injective by Lemma 3.1(iii). Hence, $\bar{\phi}$ is injective, and (i) holds.

(ii) If $\ker\phi = \{1\}$, then $\phi = \bar{\phi}$, and hence ϕ is injective by (i). On the other hand, if ϕ is injective, then ϕ_S is injective, and hence $\ker\phi = \{1\}$. So (ii) holds. □

Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes such that ϕ_S is surjective. By Example 3.10 below, ϕ may not be surjective. But if $\ker\phi$ is a finite subset of S, and ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$, then ϕ is surjective by the next lemma.

Lemma 3.5. *Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of naturally valenced association schemes such that $\ker\phi$ is a finite subset of S. Assume that ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$. Then the following hold.*

(i) *$\phi(S)$ is a closed subset of $\widetilde{S}$.*
(ii) *ϕ is injective (surjective, bijective, resp.) if and only if φ is injective (surjective, bijective, resp.).*

Proof. (i) Since $\varphi(S)$ is a closed subset of $\widetilde{S}$ by Lemma 2.3(ii), $\phi(S)$ is a closed subset of $\widetilde{S}$ by (9). So (i) holds.

(ii) Note that φ is injective if and only if $\ker_S(\varphi) = \{1\}$ by Theorem 2.6(iii). Since ϕ induces the scheme ring homomorphism φ, ϕ is injective if and only if $\ker\phi = \{1\}$ by Corollary 3.4(ii). But $\ker_S(\varphi) = \ker\phi$ by (9). Thus, ϕ is injective if and only if φ is injective.

If ϕ is surjective, then clearly φ is also surjective. Now assume that φ is surjective. Then we show that ϕ is also surjective. Clearly ϕ_S, the restriction of ϕ to S, is surjective. It remains to show that ϕ_X, the restriction of ϕ to X, is surjective. Let $\bar{\phi} : (X/T, S//T) \to (\widetilde{X}, \widetilde{S})$ be the morphism induced by ϕ.

Then for any $x \in X$, $\bar{\phi}(xT) = \phi(x)$, and for any $s \in S$, $\bar{\phi}(s^T) = \phi(s)$. Let $x \in X$. Then $X/T = (xT)(S//T)$ and $\widetilde{X} = \bar{\phi}(xT)\widetilde{S}$. For any $s^T \in S//T$, let $(xT)s^T := \{yT \in X/T \mid (xT, yT) \in s^T\}$, and $\bar{\phi}(xT)\bar{\phi}(s^T) := \{\tilde{y} \in \widetilde{X} \mid (\bar{\phi}(xT), \tilde{y}) \in \bar{\phi}(s^T)\}$. Since $\bar{\phi}$ is injective by Corollary 3.4(i), $\bar{\phi}$ induces an injective map

$$\bar{\phi}_{(xT)s^T} : (xT)s^T \longrightarrow \bar{\phi}(xT)\bar{\phi}\big(s^T\big), \qquad yT \longmapsto \bar{\phi}(yT).$$

Recall that the cardinality of the set $(xT)s^T$ is n_{s^T}, and the cardinality of the set $\bar{\phi}(xT)\bar{\phi}(s^T)$ is $n_{\phi(s)}$. But $n_{s^T} = n_{\phi(s)}$ $(< \infty)$ by Lemma 3.3(iii). So $\bar{\phi}_{(xT)s^T}$ is also surjective. Since ϕ_S is surjective, $\bar{\phi}_{S//T}$ is also surjective. So

$$\widetilde{X} = \bigcup_{s^T \in S//T} \big(\bar{\phi}(xT)\bar{\phi}\big(s^T\big)\big).$$

Thus, $\bar{\phi}_{X/T}$ must be surjective (in fact, it is bijective). Hence ϕ_X is surjective. Thus, we have proved that ϕ is surjective if and only if φ is surjective. So (ii) holds. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. If ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$, then (i) and (ii) hold by Lemma 3.3. Now assume that (i) and (ii) hold. Then we prove that ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$.

Let us first assume that ϕ is injective. Then $\ker\phi = \{1\}$, and hence the condition (ii) yields that

$$n_p = n_{\phi(p)}, \quad \forall p \in S. \tag{10}$$

For any $p, q \in S$, $\phi(pq) \subseteq \phi(p)\phi(q)$ by [Z, Lemma 5.1.1]. So

$$\sigma_{\phi(p)}\sigma_{\phi(q)} = \sum_{r \in pq} a_{\phi(p)\phi(q)\phi(r)}\sigma_{\phi(r)} + \sum_{\tilde{s} \in \phi(p)\phi(q)\setminus\phi(pq)} a_{\phi(p)\phi(q)\tilde{s}}\sigma_{\tilde{s}}.$$

Hence, from Lemma 3.1(i), (10), and [Z, Lemma 1.1.3(iv)],

$$\begin{aligned}
n_{\phi(p)}n_{\phi(q)} &= \sum_{r \in pq} a_{\phi(p)\phi(q)\phi(r)}n_{\phi(r)} + \sum_{\tilde{s} \in \phi(p)\phi(q)\setminus\phi(pq)} a_{\phi(p)\phi(q)\tilde{s}}n_{\tilde{s}} \\
&\geqslant \sum_{r \in pq} a_{pqr}n_r + \sum_{\tilde{s} \in \phi(p)\phi(q)\setminus\phi(pq)} a_{\phi(p)\phi(q)\tilde{s}}n_{\tilde{s}} \\
&= n_p n_q + \sum_{\tilde{s} \in \phi(p)\phi(q)\setminus\phi(pq)} a_{\phi(p)\phi(q)\tilde{s}}n_{\tilde{s}}.
\end{aligned}$$

But $n_{\phi(p)}n_{\phi(q)} = n_p n_q$ by (10). So we must have that $\phi(pq) = \phi(p)\phi(q)$, and $a_{pqr} = a_{\phi(p)\phi(q)\phi(r)}$, $\forall r \in S$. Therefore, if ϕ is injective, then ϕ induces a scheme ring monomorphism

$$\varphi : \mathbb{C}S \longrightarrow \mathbb{C}\widetilde{S}, \qquad \sigma_s \longmapsto \sigma_{\phi(s)}.$$

Now assume that ϕ is an arbitrary morphism. Let $T := \ker\phi$. Then by (ii) and Lemma 3.1(iii), ϕ induces a monomorphism $\bar{\phi} : (X/T, S//T) \to (\widetilde{X}, \widetilde{S})$ such that $\bar{\phi}(xT) = \phi(x)$ for any $xT \in X/T$, and $\bar{\phi}(s^T) = \phi(s)$ for any $s^T \in S//T$. Thus, by what we have just proved, $\bar{\phi}$ induces a scheme ring monomorphism

$$\bar{\varphi} : \mathbb{C}(S//T) \longrightarrow \mathbb{C}\widetilde{S}, \qquad \sigma_{s^T} \longmapsto \sigma_{\phi(s)}.$$

Since T is a normal closed subset of S by the condition (i), by Corollary 2.7 we have a scheme ring epimorphism $\pi : \mathbb{C}S \to \mathbb{C}(S//T)$ such that $\pi(\sigma_s) = (n_s/n_{s^T})\sigma_{s^T}$, for any $s \in S$. Let $\varphi := \bar{\varphi}\pi$. Then $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ is a scheme ring homomorphism, and for any $s \in S$, $\varphi(\sigma_s) = (n_s/n_{s^T})\bar{\varphi}(\sigma_{s^T}) = (n_s/n_{\phi(s)})\sigma_{\phi(s)}$. So φ is the scheme ring homomorphism induced by ϕ. Thus, we have proved that if the conditions (i) and (ii) hold, then ϕ induces a scheme ring homomorphism. This completes the proof of the theorem. □

Theorem 1.4 has a few interesting corollaries. Let (X, S), $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ an injective morphism. Then $\ker\phi = \{1_X\}$ is a normal closed subset of S. So by Theorem 1.4, ϕ induces a scheme ring monomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ if and only if for any $s \in S$, $n_s = n_{\phi(s)}$.

From Theorem 1.4 and Lemma 3.3, we have the following

Corollary 3.6. *Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a morphism such that $\ker\phi$ is a finite subset of S. Let $T = \ker\phi$, and $\bar{\phi} : (X/T, S//T) \to (\widetilde{X}, \widetilde{S})$ the morphism induced by ϕ. Then the following are equivalent.*

(i) *ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$.*
(ii) *T is a normal closed subset of S, and $\bar{\phi}$ induces a scheme ring monomorphism $\tilde{\varphi} : \mathbb{C}(S//T) \to \mathbb{C}\widetilde{S}$.*

Proof. (i) implies (ii) by Lemma 3.3(i), (ii). Now we show that (ii) implies (i). For any $s^T \in S//T$, since $\tilde{\varphi}$ is induced by $\bar{\phi}$ and $\bar{\phi}(s^T) = \phi(s)$, we see that $\tilde{\varphi}(\sigma_{s^T}) = (n_{s^T}/n_{\phi(s)})\sigma_{\phi(s)}$. But $\tilde{\varphi}$ is injective. So from the proof of Lemma 2.4, we must have that $n_{s^T} = n_{\phi(s)}$, for any $s \in S$. Thus, (i) holds by Theorem 1.4. □

The next corollary is a direct consequence of Theorem 1.4 and Corollary 3.2(ii).

Corollary 3.7. *Let (X, S) be an association scheme of finite valency, $(\widetilde{X}, \widetilde{S})$ a naturally valenced association scheme, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a morphism. Let $T = \ker\phi$, and $\bar{\phi} : (X/T, S//T) \to (\widetilde{X}, \widetilde{S})$ be the morphism induced by ϕ. Then ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ if and only if the following two conditions are satisfied.*

(i) *$\ker\phi$ is a normal closed subset of S.*
(ii) *$\bar{\phi}$ is a monomorphism, and $n_{\phi(S)} = n_S/n_{\ker\phi}$.*

An association scheme is called *commutative* if its scheme ring is commutative. Since any closed subset of a commutative association scheme is normal, the next corollary follows from Theorem 1.4 directly.

Corollary 3.8. *Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a morphism such that $\ker\phi$ is a finite subset of S. If S is commutative, then ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ if and only if for any $s \in S$, $n_{\phi(s)} = n_{s^T}$, where $T = \ker\phi$.*

Let (X, S) be an association scheme of finite valency, $(\widetilde{X}, \widetilde{S})$ a naturally valenced association scheme, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a morphism. Let $\bar{\phi} : (X/\ker\phi, S//\ker\phi) \to (\widetilde{X}, \widetilde{S})$ be the morphism induced by ϕ. If S is commutative, then by Corollary 3.7, ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ if and only if $\bar{\phi}$ is a monomorphism and $n_{\phi(S)} = n_S/n_{\ker\phi}$.

Corollary 3.9. *Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a morphism such that $\ker\phi$ is a finite subset of S. Let*

$$\bar{\phi} : (X/\ker\phi, S//\ker\phi) \to (\widetilde{X}, \widetilde{S})$$

be the morphism induced by ϕ. If ϕ_X is surjective, then ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ if and only if $\bar{\phi}$ is a monomorphism and $\ker\phi$ *is normal.*

Proof. We only need to show that if $\bar{\phi}$ is a monomorphism and $\ker\phi$ is normal, then ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$. Since ϕ_X is surjective, ϕ is surjective. Hence, $\bar{\phi}$ is also surjective. Thus, $\bar{\phi}$ is an isomorphism. Therefore, for any $s \in S$, $n_{\phi(s)} = n_{s^T}$, where $T = \ker\phi$. So ϕ induces a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$ by Theorem 1.4. □

Let (X, S) and $(\widetilde{X}, \widetilde{S})$ be naturally valenced association schemes, and let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism such that ϕ_X is surjective and $\ker\phi$ is normal. Example 3.11 below says that ϕ may not induce a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$.

Finally, let us prove Corollary 1.5.

Proof of Corollary 1.5. (i) ⇒ (ii) Let $\bar{\phi} : (X/T, S//T) \to (\widetilde{X}, \widetilde{S})$ be the morphism induced by ϕ. Then $\bar{\phi}(xT) = \phi(x)$ for any $xT \in X/T$, and $\bar{\phi}(s^T) = \phi(s)$ for any $s^T \in S//T$. By (i) and Lemma 3.1(iii), $\bar{\phi}$ is a monomorphism. Thus, for any $s \in S$, $n_{\bar{\phi}(s^T)} = n_{\phi(s)} = n_{s^T}$, and hence $\bar{\phi}$ induces a scheme ring homomorphism $\varphi : \mathbb{C}(S//T) \to \mathbb{C}\widetilde{S}$ by Theorem 1.4. So $\phi(S) = \bar{\phi}(S//T)$ is a closed subset of $\widetilde{S}$ by Lemma 3.5(i).

Let $x \in X$. Then for any $y \in X$, there exists $s \in S$ such that $(x, y) \in s$, and hence $(\phi(x), \phi(y)) \in \phi(s)$. So $\phi(y) \in \phi(x)\phi(S)$. Thus, $\phi(X) \subseteq \phi(x)\phi(S)$. Since $\bar{\phi}(X/T) = \phi(X)$ and $\bar{\phi}(S//T) = \phi(S)$, $\bar{\phi}$ induces a morphism

$$\bar{\phi}' : (X/T, S//T) \longrightarrow \big(\phi(x)\phi(S), \phi(S)_{\phi(x)\phi(S)}\big), \qquad yT \longmapsto \phi(y), \qquad s^T \longmapsto \phi(s)_{\phi(x)\phi(S)}.$$

In the following we prove that $\bar{\phi}'$ is an isomorphism. The injectivity of $\bar{\phi}_{X/T}$ forces that $\bar{\phi}'_{X/T}$ is injective. Note that for any $s \in S$, $n_{\phi(s)} = n_{\phi(s)_{\phi(x)\phi(S)}}$ by the definition of the subscheme $(\phi(x)\phi(S), \phi(S)_{\phi(x)\phi(S)})$. (Also by [BI, Chapter II, Theorem 9.3, p. 137].) So for any $s^T \in S//T$, $n_{\bar{\phi}'(s^T)} = n_{\phi(s)} = n_{s^T}$. Hence, $\bar{\phi}'_{S//T}$ is injective by Lemma 3.1(iii). Thus, $\bar{\phi}'$ is a monomorphism, and induces a scheme ring homomorphism $\varphi : \mathbb{C}(S//T) \to \mathbb{C}(\phi(S)_{\phi(x)\phi(S)})$ by Theorem 1.4. Since the restriction of $\bar{\phi}'$ to $S//T$ is surjective, φ is surjective. Hence $\bar{\phi}'$ is surjective by Lemma 3.5(ii). Thus, $\bar{\phi}'$ is an isomorphism, and for any $x \in X$,

$$(X/T, S//T) \cong \big(\phi(x)\phi(S), \phi(S)_{\phi(x)\phi(S)}\big).$$

So (ii) holds.

(ii) ⇒ (i) Trivial. □

Let (X, S) be an association scheme of finite valency, $(\widetilde{X}, \widetilde{S})$ a naturally valenced association scheme, and $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ a morphism. Let $\bar{\phi} : (X/\ker\phi, S//\ker\phi) \to (\widetilde{X}, \widetilde{S})$ be the morphism induced by ϕ. Then by Corollaries 3.2(ii) and 1.5, $\bar{\phi}$ is a monomorphism and $n_{\phi(S)} = n_S/n_{\ker\phi}$ if and only if $\phi(S)$ is a closed subset of $\widetilde{S}$ and $(X/\ker\phi, S//\ker\phi) \cong (\phi(x)\phi(S), \phi(S)_{\phi(x)\phi(S)})$ for any $x \in X$.

Let $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ be a morphism of association schemes such that $\ker\phi$ is a finite subset of S and $\phi(S)$ is a closed subset of $\widetilde{S}$. The next example indicates that it is not necessarily true that $(X/\ker\phi, S//\ker\phi) \cong (\phi(x)\phi(S), \phi(S)_{\phi(x)\phi(S)})$ for some $x \in X$.

Example 3.10. Let $X = \{x_i \mid 1 \leqslant i \leqslant 6\}$, and $S := \{s_1 = 1_X, s_2, s_3, s_4\}$, where

$$s_2 = \{(x_1, x_2), (x_2, x_1), (x_3, x_5), (x_4, x_6), (x_5, x_3), (x_6, x_4)\},$$
$$s_3 = \{(x_1, x_3), (x_1, x_4), (x_2, x_5), (x_2, x_6), (x_3, x_1), (x_3, x_4),$$
$$(x_4, x_1), (x_4, x_3), (x_5, x_2), (x_5, x_6), (x_6, x_2), (x_6, x_5)\},$$
$$s_4 = (X \times X) \setminus (s_1 \cup s_2 \cup s_3).$$

Then (X, S) is an association scheme.

For any group G, we know that $G^\tau := \{g^\tau \mid g \in G\}$ is a thin association scheme on G, where $g^\tau := \{(h,k) \in G \times G \mid h^{-1}k = g\}$, $\forall g \in G$. In particular, let $G := \{g_1 = 1, g_2, g_3, g_4\}$ be the Klein four-group. Then the map

$$\phi : G \cup G^\tau \longrightarrow X \cup S, \qquad g_1 \longmapsto x_1, \qquad g_2 \longmapsto x_2, \qquad g_3 \longmapsto x_3, \qquad g_4 \longmapsto x_5,$$
$$g_i^\tau \longmapsto s_i, \quad 1 \leqslant i \leqslant 4$$

is a morphism from (G, G^τ) to (X, S) such that $\ker\phi = \{g_1^\tau\}$ and $\phi(G^\tau) = S$. So $\phi(G^\tau)$ is a closed subset of S. However, $n_{\phi(G^\tau)} = 6 \neq 4 = n_{G^\tau}/n_{\ker\phi}$, and

$$\left(G/\ker\phi, G^\tau//\ker\phi\right) \cong \left(G, G^\tau\right) \ncong (X, S) = \left(\phi(g_i)\phi\left(G^\tau\right), \phi\left(G^\tau\right)_{\phi(g_i)\phi(G^\tau)}\right), \quad \forall g_i \in G.$$

The next example is due to the referee. It has been cited several times in this section to clarify the significance of the condition $n_{s^T} = n_{\phi(s)}$ in Theorem 1.4 and its consequences.

Example 3.11. Let (X, S) be a naturally valenced association scheme. Assume that (X, S) is not symmetric. That is, there exists $s \in S$ such that $s^* \neq s$. Let $(\widetilde{X}, \widetilde{S})$ be the symmetrization of (X, S). Thus, $\widetilde{X} = X$, and $\widetilde{S} = \{\tilde{s} \mid \tilde{s} = s \cup s^*,\ \text{all } s \in S\}$. It is well known that $(\widetilde{X}, \widetilde{S})$ is a naturally valenced association scheme. Define $\phi : (X, S) \to (\widetilde{X}, \widetilde{S})$ by

$$\phi(x) = x, \quad \text{for all } x \in X; \quad \text{and} \quad \phi(s) = s \cup s^*, \quad \text{for all } s \in S.$$

Then ϕ is a morphism, ϕ_X is bijective, ϕ_S is surjective, but ϕ_S is not injective (because $\phi(s) = \phi(s^*)$ for all $s \in S$). Thus, ϕ is not a monomorphism. Note that $\ker\phi = \{1\}$. So $\ker\phi$ is a normal closed subset of S. For any $s \in S$,

$$n_{\phi(s)} = \begin{cases} n_s, & \text{if } s = s^*; \\ 2n_s, & \text{if } s \neq s^*. \end{cases}$$

Let $T = \ker\phi$. Then for any $s \in S$, $n_{s^T} = n_s$. Thus, $n_{s^T} \neq n_{\phi(s)}$ for any $s \in S$ such that $s \neq s^*$. Let $\bar{\phi} : (X/T, S//T) \to (\widetilde{X}, \widetilde{S})$ be the morphism induced by ϕ. Then $\bar{\phi} = \phi$, and hence $\bar{\phi}$ is not a monomorphism. If S has finite valency, then $n_{\phi(S)} = n_S = n_S/n_T$. Finally, ϕ does not induce a scheme ring homomorphism $\varphi : \mathbb{C}S \to \mathbb{C}\widetilde{S}$, and $(X, S) \ncong (\widetilde{X}, \widetilde{S})$.

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